Intégrales (suite)

$$F \xrightarrow{D} f \xrightarrow{I} F$$
$$f \xrightarrow{D} f' \xrightarrow{I} f$$

Ex 3 : $\frac{u'}{u}$ ou $\frac{u'}{u^2}$ ou $u' e^u$

$\frac{u'}{u} \to \ln(u)$ ⚠ $u > 0$

$\frac{u'}{u^2} \to \frac{-1}{u}$ ⚠ $u \neq 0$

$u' e^u \to e^u$

$$f_3(x) = 3x\, e^{3x^2+1} = \frac{1}{2} \times \boxed{6x\, e^{3x^2+1}} \to F_3(x) = \frac{1}{2} \times e^{3x^2+1} = \frac{1}{2} e^{3x^2+1} + k$$

$u(x) = 3x^2 + 1$

$u'(x) = 3 \times 2x = 6x$

$u' e^u = 6x\, e^{3x^2+1}$

$$f_4(x) = \frac{e^x}{(e^x+1)^2} = \frac{u'(x)}{[u(x)]^2} \qquad F_4(x) = \frac{-1}{e^x+1} + k$$

$u(x) = e^x + 1$
$u'(x) = e^x$

$$f_5(x) = \frac{e^x}{e^x+3} = \frac{u'(x)}{u(x)} \qquad F_5(x) = \ln(e^x+3) + k$$

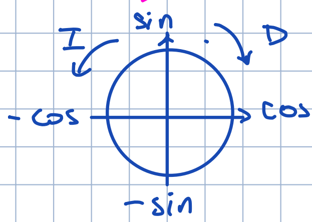
$u(x) = \underbrace{e^x}_{>0} + 3 > 0$ pour tout $x \in \mathbb{R}$

$u'(x) = e^x$

$$f_6(x) = \cos(x)\, e^{\sin(x)} = u'(x)\, e^{u(x)} \qquad F_6(x) = e^{\sin(x)} + k$$

$u(x) = \sin(x)$

$u'(x) = \cos(x)$



Ex 4 :

$$f(x) = \frac{x}{x^2+4} = \frac{u'(x)}{u(x)} = \frac{1}{2} \times \boxed{\frac{2x}{x^2+4}} \qquad F(x) = \frac{1}{2} \times \boxed{\ln(x^2+4)} + k$$

$u(x) = x^2 + 4$
$u'(x) = 2x$

$u(x) = \underbrace{x^2}_{\geq 0} + 4 > 0$

Exercice : $f(x) = \dfrac{x^2}{(x^3+1)^2} = \dfrac{u'(x)}{[u(x)]^2} = \dfrac{1}{3} \times \boxed{\dfrac{3x^2}{(x^3+1)^2}}$ $\qquad F(x) = \dfrac{1}{3} \times \boxed{\left(\dfrac{-1}{x^3+1}\right)} + k$

pour $x \neq -1$

$\dfrac{u'(x)}{[u(x)]^2} \longrightarrow \dfrac{-1}{u(x)}$

$u(x) = x^3 + 1$

$u'(x) = 3x^2$

$u \neq 0$

$x^3+1 = 0$

$x = -1$

Exercice : $f(x) = (x^4+2)\, e^{x^5+10x+3} = u'(x)\, e^{u(x)} = \dfrac{1}{5}(5x^4+10)\, e^{x^5+10x+3}$

$u(x) = x^5 + 10x + 3$

$u'(x) = 5x^4 + 10 = 5(x^4+2)$ $\qquad F(x) = \dfrac{1}{5}\, e^{x^5+10x+3} + k$

Exercice : $f(x) = \dfrac{2e^x}{(e^x)^2} = \dfrac{u'(x)}{(u(x))^2} = 2 \times \dfrac{e^x}{(e^x)^2}$ $\qquad F(x) = 2 \times \dfrac{-1}{e^x} = \dfrac{-2}{e^x} + k$

$u(x) = e^x$

$u'(x) = e^x \neq 0$

Exercice : $f(x) = \dfrac{x+1}{2x^2+4x+3} = \dfrac{u'(x)}{u(x)} = \dfrac{1}{4} \times \dfrac{4(x+1)}{2x^2+4x+3}$

$u(x) = 2x^2 + 4x + 3$

$u'(x) = 2 \times 2x + 4 = 4x + 4 = 4(x+1)$

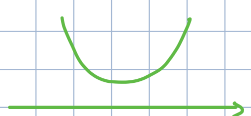
⚠ $u(x) > 0$? $\quad 2x^2+4x+3$

$a=2 \quad b=4 \quad c=3$

$\Delta = b^2 - 4ac = 4^2 - 4 \times 2 \times 3$

$= 16 - 24$

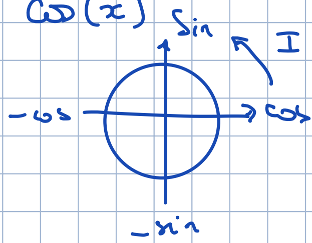
$= -8 < 0$



donc $u(x) > 0$ sur $\mathbb{R}$

$F(x) = \dfrac{1}{4} \times \ln(2x^2+4x+3) + k$

Exercice 5 : 1) $f(x) = \cos(x)$ $\qquad F(x) = \sin(x) + k \; , k \in \mathbb{R}$



$F(x) = \sin(x) + k$ s'annule en $x = \frac{\pi}{2}$

$F\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) + k = 0$

$1 + k = 0$

$k = -1$ donc $\boxed{F(x) = \sin(x) - 1}$

2) $f(x) = e^{3x+1} = u'(x)e^{u(x)} = \frac{1}{3} \times 3\boxed{e^{3x+1}}$ $\qquad F(x) = \frac{1}{3} \times \boxed{e^{3x+1}} + k$

$u(x) = 3x+1$

$u'(x) = 3$

$F(x) = \frac{1}{3} e^{3x+1} + k$

$\left.\begin{array}{l} F(-1) = 0 \\ F(-1) = \frac{1}{3} e^{3\times(-1)+1} + k = \frac{1}{3} e^{-2} + k \end{array}\right\}$

$\frac{1}{3} e^{-2} + k = 0$

$k = -\frac{1}{3} e^{-2}$

donc $F(x) = \frac{1}{3} e^{3x+1} - \frac{1}{3} e^{-2}$

3) $F(\sqrt{\ln 2}) = 1.$

$f(x) = x e^{-x^2} = u'(x) e^{u(x)} = \frac{-1}{2} \times \left(-2x e^{-x^2}\right)$ $\qquad F(x) = -\frac{1}{2} e^{-x^2} + k$

$u(x) = -x^2$

$u'(x) = -2x$

$F\left(\sqrt{\ln(2)}\right) = -\frac{1}{2} e^{-\left(\sqrt{\ln(2)}\right)^2} + k$

$= -\frac{1}{2} e^{-\ln(2)} + k$

$= -\frac{1}{2} e^{\ln\left(\frac{1}{2}\right)} + k$

$= -\frac{1}{2} \times \frac{1}{2} + k$

donc $F(\sqrt{\ln(2)}) = -\frac{1}{4} + k$ d'où $-\frac{1}{4} + k = 1$

$k = 1 + \frac{1}{4} = \frac{5}{4}$

donc $F(x) = -\frac{1}{2} e^{-x^2} + \frac{5}{4}$

II) Intégrales

Définition : on appelle intégrale de f sur $[a;b] \subset \mathbb{R}$

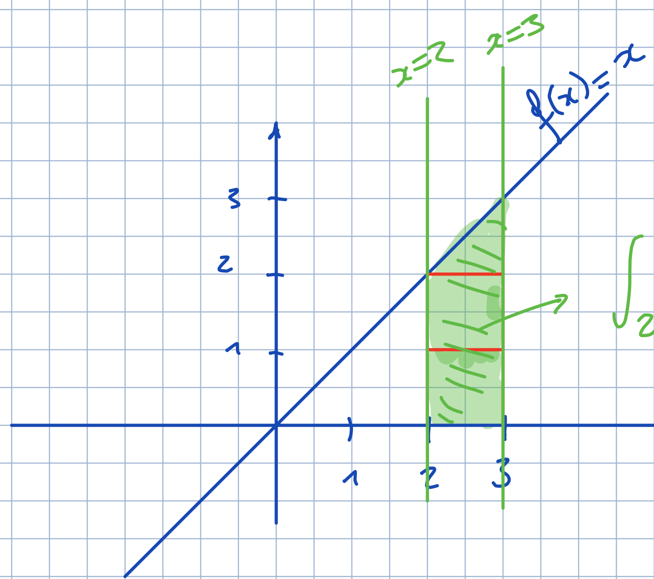
et on note $$\int_a^b f(x)\,dx = F(b) - F(a)$$

avec F primitive de f.

Exemple : $$\int_2^3 x\,dx = F(3) - F(2) = \frac{3^2}{2} - \frac{2^2}{2} = \frac{9}{2} - \frac{4}{2} = \frac{5}{2} = 2{,}5$$

$f(x) = x$ $\quad F(x) = \frac{x^2}{2}$

Interprétation graphique

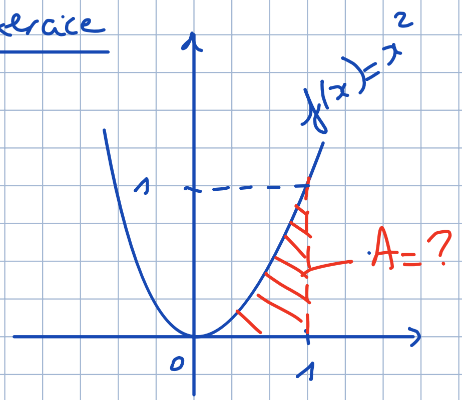


$\int_2^3 f(x)\,dx$ = Aire entre la courbe $\mathcal{C}_f$

l'axe (Ox) et

les droites $x=2$ et $x=3$

Exercice



$$A = \int_0^1 x^2\,dx = F(1) - F(0) = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

$f(x) = x^2 \to F(x) = \frac{x^3}{3}$

Ex7:

a) $\int_{-1}^{1} t^2+4t+3 \, dt = F(1) - F(-1)$

$f(t) = t^2+4t^1+3$

$F(t) = \frac{t^3}{3} + 4\frac{t^2}{2} + 3t$

$F(t) = \frac{t^3}{3} + 2t^2 + 3t$

x^n

$\downarrow$

$\frac{x^{n+1}}{n+1}$

$= \frac{1^3}{3} + 2\times1^2 + 3\times1 - \left(\frac{(-1)^3}{3} + 2\times(-1)^2 + 3\times(-1)\right)$

$= \frac{1}{3} + 2 + 3 - \left(\frac{-1}{3} + 2 - 3\right)$

$= \frac{1}{3} + 5 + \frac{1}{3} - 2 + 3$

$= \frac{2}{3} + \frac{6\times3}{1\times3}$

$= \frac{20}{3}$

Correction:

Ex1: /4,5

$F_1(x) = \frac{3x^2}{2} + 2x$

$F_2(x) = -\frac{4x^2}{2} + x$
$= -2x^2 + x$

$F_3(x) = 3\times\frac{x^3}{3}$
$= x^3$

$F_4(x) = \frac{x^3}{3} + x$

$F_5(x) = \frac{x^6}{6}$

$F_6(x) = \sin(x)$

$F_7(\cancel{x}) e^x$

$F_8(x) = -\frac{1}{x}$

$F_9(x) = \ln(x)$

Ex2:

$F_1(x) = 3\frac{x^3}{3} - 2\frac{x^2}{2} + x$
$= x^3 - x^2 + x$

$F_2(x) = \frac{x^6}{6} - \frac{x^5}{5} + \frac{x^4}{4} + 3x$

$F_3(x) = \frac{5}{2}\frac{x^8}{8} = \frac{5}{16}x^8$

$F_4(x) = \frac{e^{3x}}{3}$

$F_5(x) = \frac{5e^{2x}}{2}$

$F_6(x) = \frac{2e^{2x+1}}{2}$
$= e^{2x+1}$

$F_7(x) = \sin(5x+1)$

$F_8(x) = e^{x^2+3}$

$F_9(x) = \frac{\sin(2x)}{2}$

$F_{10}(x) = \ln(x+1)$, $x > -1$

$F_{11}(x) = \frac{-1}{x+1}$, $x \neq -1$

$F_{12}(x) = \frac{1}{4}\ln(x^4+1)$

$x \in \mathbb{R}$

Ex3:

$$I_1 = \int_0^1 2x\,dx = 1^2 - 0^2 = 1 \quad 0{,}5$$

$0{,}5 \quad F(x) = x^2$

$$I_2 = \int_0^1 x^2\,dx = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

$$F(x) = \frac{x^3}{3}$$

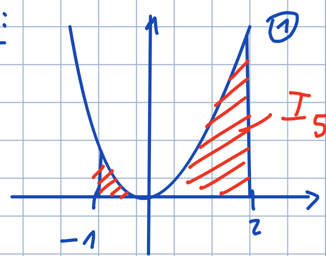
$$I_3 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x)\,dx = -\cos\left(\frac{\pi}{2}\right) - \left(-\cos\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \simeq 0{,}785$$

$$F(x) = -\cos(x)$$

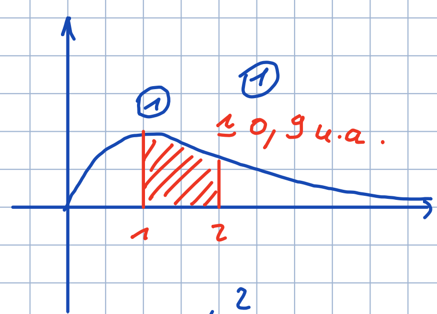
$$I_4 = \int_{-0{,}5}^{1} e^{2x+1}\,dx = \frac{e^{2\times1+1}}{2} - \frac{e^{2\times(-0{,}5)+1}}{2} = \frac{e^3}{2} - \frac{1}{2} \simeq 9{,}543$$

$$F(x) = \frac{e^{2x+1}}{2}$$

Ex4:



① $I_5 \simeq 3{,}5$ u.a. ①



$0{,}5$

$$I_5 = \frac{2^3}{3} + \frac{1^3}{3} = \frac{9}{3} = 3 \text{ u.a.}$$

① $$I_6 = \int_1^2 \frac{2x}{x^2+1}\,dx = \ln(2^2+1) - \ln(1^2+1)$$

$$= \ln(5) - \ln(2)$$

$$= \ln\left(\frac{5}{2}\right)$$

$$\simeq 0{,}916$$

$$F(x) = \ln(x^2+1)$$

$0{,}5$